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Strong Large Deviation and Local Limit Theorems

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Abstract

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Most large deviation results give asymptotic expressions to $\log P(Y_n \geq x_n)$ where the event $\{Y_n \geq x_n\}$ is a large deviation event, that is , its probability goes to zero exponentially fast. ~~We~~ The authors refer to such results as weak large deviation results. ~~In~~ this paper ~~we~~ obtains strong large deviation results for arbitrary random variables $\{Y_n\}$, that is, ~~we~~ it obtains asymptotic expressions for $P(Y_n \geq x_n)$ where $\{Y_n \geq x_n\}$ is a large deviation event. These strong large deviation results are obtained for lattice valued and nonlattice valued random variables and require some conditions on their moment generating functions.

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A result that gives the limit of the average probability that Y_n lies in an interval $2h/b_n$ around the point Y_n, where $h > 0$, b_n $\cancel{\to \infty}$ and $y_n \to y^*$, is referred to as a local limit result for $\{Y_n\}$. ~~In~~ this paper ~~we~~ obtains local limit theorems for arbitrary random variables based on easily verifiable conditions on their characteristic functions. These local limit theorems play a major role in the proofs of the strong large deviation results of this paper.

~~We illustrate~~ these results are illustrated with two typical applications.

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1. Introduction

The establishment of a limit distribution for a sequence of random variables $\{Y_n, n \geq 1\}$ provides an approximation to $P(Y_n \leq x)$. However, there are other aspects relating to the distribution of Y_n for which one often desires an approximation. This could be $P(Y_n \geq x_n)$, known in the literature as a large deviation, especially when it tends to zero exponentially fast. Another example is $k_n(x_n)$, the probability density function of Y_n at x_n. The term, a large deviation local limit result for Y_n, is used when an asymptotic expression is established for $k_n(x_n)$ and x_n is in the range of a large deviation for Y_n. Still another example is the average probability that Y_n gives to an interval of length $2h/b_n$ around a point y_n, where $h > 0$ and $b_n \to \infty$. An asymptotic expression for $(b_n/2h)\, P(|Y_n - y_n| < h/b_n)$ will be referred to as a local limit result for Y_n. We say that B_n is an asymptotic expression for A_n, in symbols $A_n \sim B_n$, if $A_n/B_n \to 1$.

The theory of large deviations for sums of i.i.d. random variables and its many generalizations has a long history, see for instance Cramer(1938), Chernoff(1952), Ellis(1984), Varadhan(1984) etc. However, most of these results give asymptotic expressions for $\log P(Y_n \geq x_n)$ and so we choose to call them weak large deviation results. For arbitrary random variables Y_n, this paper gives asymptotic expressions for $P(Y_n \geq x_n)$, which we call strong large deviation results. These results are found in Theorems 3.1 and 4.3, which impose conditions on the moment generating function (m.g.f.) of Y_n. These extend the well-known strong large deviation results for sums of i.i.d. random variables due to Bahadur and Ranga Rao(1960).

The proofs of Theorem 3.1 and 4.3 depend on the local limit results for Y_n. These are established first in this paper in Theorems 2.1, 2.4 and 4.1. and they are in the spirit

of Feller(1967) wherein can be found some of the first local limit results for sums of i.i.d. random variables. DeHaan and Resnick(1982) established local limit results for extreme values and Jain and Pruitt(1985) for sums of triangular arrays of i.i.d. random variables. The local limit results in this paper apply to arbitrary random variables Y_n and require some easily verifiable conditions on their characteristic functions.

We illustrate our general results with two applications in Section 5. The first application is a local limit result for sums of dependent random variables given by a general model considered in Chaganty and Sethuraman(1986a). The second application is a strong large deviation result for the Wilcoxon signed- rank statistic under the null hypothesis.

We do not study large deviation local limit results in this paper. We have obtained such results for arbitrary random variables in Chaganty and Sethuraman(1985) for one-dimensional random variables and in Chaganty and Sethuraman(1986b) for multidimensional random variables.

2. Local Limit Theorems

Let $\{Y_n, n \geq 1\}$ be an arbitrary sequence of random variables which converge to Y in distribution. We do not assume that Y_n has a probability density function (p.d.f.). Let $\{y_n\}$ and $\{b_n\}$ be two sequences of real numbers such that $y_n \to y^*$ and $b_n \to \infty$. By a local limit theorem for Y_n, we mean that if $h > 0$, the average probability that Y_n assigns to an interval of length $2h/b_n$ around y_n converges to the p.d.f. of Y at y^*. This is the spirit under which local limit theorems have been studied for normalized sums of i.i.d. random variables by Feller(1967), for normalized extreme values in DeHaan and Resnick(1982) and for normalized triangular arrays of i.i.d. random variables in Jain and Pruitt(1985). This section is devoted to local limit theorems for arbitrary random variables Y_n. The main result is the theorem stated below.

Theorem 2.1. Let $\{Y_n, n \geq 1\}$ be a sequence of nonlattice valued random variables which converge to Y in distribution. Let $\hat{f}_n$ be the characteristic function (c.f.) of Y_n for $n \geq 1$ and let $\hat{f}$ be the c.f. of Y. Suppose that there are sequences $\{d_n\}, \{b_n\}$ with $d_n \to \infty$, $b_n \to \infty$ and an integrable function $f^*(t)$ such that

$$\sup_n |\hat{f}_n(t)|\, I\big(|t| < d_n\big) \leq f^*(t) \tag{2-1}$$

for each t, and

$$\sup_{|t| \geq d_n} |\hat{f}_n(t)| = \theta_n = o(1/b_n) \tag{2-2}$$

as $n \to \infty$.

Then the random variable Y possesses a bounded p.d.f. f. Let $h > 0$ and $y_n \to y^*$ as $n \to \infty$. Then

$$\frac{b_n}{2h} P\big(|Y_n - y_n| < h/b_n\big) \to f(y^*) \tag{2-3}$$

as $n \to \infty$. Furthermore, there exists a finite constant M and an integer n_h such that

(2-4)
$$\sup_y \left[\frac{b_n}{2h} P\big(|Y_n - y| < h/b_n\big)\right] \le M$$

for $n \ge n_h$.

Proof. Since $\hat{f}_n(t) \to \hat{f}(t)$ pointwise and $d_n \to \infty$, condition (2-1) implies that $\hat{f}$ is integrable and hence Y possesses a bounded p.d.f. f. In view of condition (2-2) we can find a sequence $\{\lambda_n\}$ satisfying

(2-5)
$$\lambda_n/b_n \to \infty \text{ and } \lambda_n\theta_n \to 0$$

as $n \to \infty$. We now introduce two distribution functions U_n, V_n with corresponding p.d.f.'s u_n, v_n and c.f.'s $\hat{u}_n$, $\hat{v}_n$ as defined below, to obtain the important identity (2-13):

(2-6)
$$u_n(x) = \begin{cases} \frac{b_n}{2h} & \text{for } |x| < h/b_n \\ 0 & \text{otherwise.} \end{cases}$$

(2-7)
$$\hat{u}_n(t) = \frac{\sin(ht/b_n)}{(ht/b_n)},$$

(2-8)
$$v_n(y) = \frac{\lambda_n}{2\pi}\left[\frac{\sin(\lambda_n y/2)}{(\lambda_n y/2)}\right]^2, \qquad \text{and}$$

(2-9)
$$\hat{v}_n(t) = \begin{cases} 1 - \frac{|t|}{\lambda_n} & \text{if } |t| \le \lambda_n \\ 0 & \text{otherwise.} \end{cases}$$

Let F_n be the distribution function (d.f.) of Y_n, and let $G_n = F_n * U_n$, $M_n = G_n * V_n$ where * denotes the convolution operation. Then the p.d.f.'s g_n, m_n of G_n and M_n are given by

(2-10)
$$g_n(x) = \frac{b_n}{2h}\int_{x-h/b_n}^{x+h/b_n} dF_n(y) = \frac{b_n}{2h} P\big(|Y_n - x| < h/b_n\big)$$

and

(2-11)
$$m_n(x) = \int_{-\infty}^{\infty} g_n(x-y)v_n(y)\,dy.$$

Since the c.f. $\hat{m}_n(t)$ of M_n, which is equal to $\hat{f}_n(t)\hat{u}_n(t)\hat{v}_n(t)$, vanishes outside of $[-\lambda_n, \lambda_n]$, the inversion formula gives

$$m_n(x) = \frac{1}{2\pi} \int_{-\lambda_n}^{\lambda_n} \exp(-itx)\hat{m}_n(t)\,dt \tag{2-12}$$

Substituting $x = y_n$, we get

$$\begin{aligned} &\frac{b_n}{2h} \int_{-\infty}^{\infty} P(|Y_n - y_n + y| < h/b_n) v_n(y)\,dy \\ &\quad = \frac{1}{2\pi} \int_{-\lambda_n}^{\lambda_n} \exp(-ity_n)\hat{m}_n(t)\,dt \\ &\quad = A_n \quad \text{(say)}. \end{aligned} \tag{2-13}$$

Relation (2-13) is the starting point of the main part of this proof and it relates $P(|Y_n - y_n| < h/b_n)$ to the integrable c.f. $\hat{m}_n(t)$. We first show that A_n which appears in (2-13) converges to $f(y^*)$ and that it is bounded above and below by $\frac{b_n}{2h} P(|Y_n - y_n| < (h \pm \eta)/b_n)$ for small $\eta > 0$, which then establishes (2-3). Notice that

$$\left| \frac{1}{2\pi} \int_{d_n \le |t| < \lambda_n} \exp(-ity_n)\,\hat{m}_n(t)\,dt \right| \le \frac{\lambda_n \theta_n}{2\pi} \to 0 \tag{2-14}$$

and

$$\begin{aligned} \frac{1}{2\pi} \int_{0 \le |t| < d_n} \exp(-ity_n)\,\hat{m}_n(t)\,dt &\to \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ity^*)\hat{f}(t)\,dt \\ &= f(y^*) \end{aligned} \tag{2-15}$$

from condition (2-1), the bounded convergence theorem and the inversion formula. Hence

$$A_n \to f(y^*). \tag{2-16}$$

Now, using (2-1) and (2-14) we get our first upper bound for A_n:

$$A_n \le \frac{\lambda_n \theta_n}{2\pi} + \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(t)\,dt. \tag{2-17}$$

Fix $\eta > 0$. We get a lower bound for A_n as follows:

$$\begin{aligned} A_n &\ge \frac{b_n}{2h} P(|Y_n - y_n| < (h - \eta)/b_n) \int_{|y| < \eta/b_n} v_n(y)\,dy \\ &\ge \frac{b_n}{2h} P(|Y_n - y_n| < (h - \eta)/b_n) \left[1 - \frac{4b_n}{\pi \lambda_n \eta}\right]. \end{aligned} \tag{2-18}$$

Combining (2-17) and (2-18) we get

$$\frac{b_n}{2h} P\big(|Y_n - y_n| < (h-\eta)/b_n\big) \left[1 - \frac{4b_n}{\pi\lambda_n\eta}\right] \le \frac{\lambda_n\theta_n}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(t)\,dt. \tag{2-19}$$

Substituting h by $2h$ and η by h, we get

$$\frac{b_n}{2h} P\big(|Y_n - y_n| < h/b_n\big) \left[1 - \frac{4b_n}{\pi\lambda_n h}\right] \le \frac{\lambda_n\theta_n}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(t)\,dt. \tag{2-20}$$

Since $\lambda_n\theta_n \to 0$, $b_n/\lambda_n \to 0$, we can find an integer n_h so that

$$\sup_{n\ge n_h}\left[\frac{b_n}{2h} P\big(|Y_n - y_n| < h/b_n\big)\right] \le M \tag{2-21}$$

where

$$M = \frac{4}{\pi}\int_{-\infty}^{\infty} f^*(t)\,dt,$$

uniformly in y_n. This proves assertion (2-4). Now for any $\eta > 0$, we can obtain another upperbound for A_n when $n \ge n_h$ by using (2-21).

$$\begin{aligned} A_n &= \frac{b_n}{2h}\int_{-\infty}^{\infty} P\big(|Y_n - y_n + y| < h/b_n\big) v_n(y)\,dy \\ &\le \frac{b_n}{2h} P\big(|Y_n - y_n| < (h+\eta)/b_n\big) + M\int_{|y|>\eta/b_n} v_n(y)\,dy \\ &\le \frac{b_n}{2h} P\big(|Y_n - y_n| < (h+\eta)/b_n\big) + \frac{4Mb_n}{\pi\lambda_n\eta}. \end{aligned} \tag{2-22}$$

Thus, from (2-16), (2-18) and (2-22) we get that

$$\begin{aligned} &\limsup_n \frac{b_n}{2h} P\big(|Y_n - y_n| < (h-\eta)/b_n\big) \\ &\qquad \le f(y^*) \le \liminf_n \frac{b_n}{2h} P\big(|Y_n - y_n| < (h+\eta)/b_n\big) \end{aligned} \tag{2-23}$$

This implies that

$$\begin{aligned} \frac{(h-\eta)}{h} f(y^*) &\le \liminf_n \frac{b_n}{2h} P\big(|Y_n - y_n| < h/b_n\big) \\ &\le \limsup_n \frac{b_n}{2h} P\big(|Y_n - y_n| < h/b_n\big) \\ &\le \frac{(h+\eta)}{h} f(y^*). \end{aligned} \tag{2-24}$$

Since $\eta > 0$ is arbitrary, this establishes (2-3). $\diamond$

Theorem 2.1 is a local limit theorem for the average probability that Y_n assigns to intervals of length $2h/b_n$, where $b_n \to \infty$. Conditions (2-1) and (2-2) of Theorem 2.1 require that the c.f. $\hat{f}_n$ of Y_n be bounded by an integrable function f^* on $[-d_n, d_n]$ and goes to zero at a suitable rate outside $[-d_n, d_n]$, where $d_n \to \infty$. There is a trade off on how large d_n and b_n should be and the rate at which $\hat{f}_n$ should go to zero outside $[-d_n, d_n]$. Remarks 2.2, 2.3 and Theorem 2.4 explore the tradeoffs.

Remark 2.2. Suppose that there exists an integrable function f^* such that $f^*(t) \to 0$ as $|t| \to \infty$ and

(2-25)
$$\sup_{n \geq 1} |f_n(t)| \leq f^*(t)$$

for all t. Then for any sequence $\{b_n\}$ with $b_n \to \infty$, we can find an sequence $\{d_n\}$ such that $d_n \to \infty$ and conditions (2-1) and (2-2) are satisfied. Thus when (2-25) holds, the conclusions (2-3) and (2-4) of Theorem 2.1 hold for every sequence $\{b_n\}$ with $b_n \to \infty$.

The above remark is used in Example 5.1 of Section 5.

Remark 2.3. The conclusions (2-3) and (2-4) of Theorem 2.1 hold if we replace condition (2-2) by

(2-26)
$$\int_{d_n}^{\lambda_n} |\hat{f}_n(t)|\, dt \to 0 \quad \text{as } n \to \infty,$$

for some sequence of real numbers $\{\lambda_n\}$, such that $\lambda_n \to \infty$, and $\lambda_n / b_n \to \infty$ as $n \to \infty$.

Theorem 2.4, stated below, shows that we can relax condition (2-2) and still obtain the conclusions (2-3) and (2-4) for sequences $\{b_n\}$ such that $d_n / b_n \to \infty$ as $n \to \infty$.

Theorem 2.4. Let $\{Y_n,\, n \geq 1\}$ be a sequence of nonlattice valued random variables which converge in distribution to Y. Assume that condition (2-1) of Theorem 2.1 holds for some

sequence of real numbers $\{d_n\}$ with $d_n \to \infty$. Let $\{b_n\}$ be a sequence of real numbers such that $b_n \to \infty$ and $d_n/b_n \to \infty$, that is b_n diverges to ∞ slower than d_n. Let $h > 0$ and $y_n \to y^*$ as $n \to \infty$. Then the conclusions (2-3) and (2-4) of Theorem 2.1 hold.

Proof. Let $\lambda_n = d_n$, then condition (2-26) trivially holds. Thus Theorem 2.4 follows from Theorem 2.1 and Remark 2.3. $\diamond$

The next theorem provides a convienent way to verify condition (2-1) of Theorem 2.1. In Lemma 3.2 of Section 3 we use Theorem 2.1 and this method of verification of condition (2-1), in the midst of our proof of a strong large deviation theorem for arbitrary random variables T_n.

Theorem 2.5. Let $\{Y_n, n \geq 1\}$ be a sequence of random variables with c.f.'s $\{\hat{f}_n(t), n \geq 1\}$. Let $\{d_n\}$ be a sequence of real numbers such that $d_n \to \infty$. Let $g_n(t) = d_n^{-2} \log |\hat{f}_n(d_n t)|$ be a well defined function of t and twice differentiable in a neighborhood of the origin. Suppose that there exists an $\delta > 0$, $\alpha > 0$ such that for $|t| < \delta$,

$$-g_n''(t) \geq \alpha \tag{2-27}$$

for all $n \geq 1$. Then condition (2-1) of Theorem 2.1 is satisfied with d_n replaced by δd_n.

Proof. An application of Taylor's theorem yields for $|t| < \delta$,

$$\begin{aligned} g_n(t) &= g_n(0) + t\, g_n'(0) + \frac{t^2}{2} g_n''(\xi_n) \\ &= \frac{t^2}{2}\, g_n''(\xi_n) \\ &\leq -\frac{\alpha t^2}{2}, \end{aligned} \tag{2-28}$$

where ξ_n is such that $|\xi_n| < |t| < \delta$. Therefore for $|t| < \delta d_n$,

$$g_n(t/d_n) \leq -\frac{\alpha t^2}{2 d_n^2}. \tag{2-29}$$

Thus for $|t| < \delta d_n$, we have for all $n \geq 1$,

$$\begin{aligned} |\hat{f}_n(t)| &= \exp(d_n^2(g_n(t/d_n))) \\ &\leq \exp(-\alpha t^2/2), \end{aligned} \tag{2-30}$$

which is an integrable function. This completes the proof of the theorem. $\diamond$

Remark 2.6. It may seem that the restriction of nonlattice random variables appears only in the statements of Theorem 2.1 but not in the proofs. A close look at condition (2-2) shows that it cannot hold if Y_n is lattice valued. The restriction to nonlattice random variables was therefore made more prominent in the statement of the Theorem 2.1 rather than tuck it away in condition (2-2). We treat the case of lattice valued random variables in Section 4.

3. Strong Large Deviation Theorems

Large deviation results for arbitrary sequences of random variables, $\{T_n, n \geq 1\}$, obtain asymptotic expressions for $\log P(T_n/a_n > m_n)$ where the event $\{T_n/a_n > m_n\}$ represents a large deviation. A number of authors, including Seivers(1969), Steinebach(1978), Ellis(1984) have obtained such results under suitable conditions on the m.g.f. of T_n. In this section we obtain strong large deviation limit theorems for T_n, i.e., asymptotic expressions for $P(T_n/a_n > m_n)$. Similar results have been obtained before when T_n is the sum of i.i.d. random variables by Bahadur and Ranga Rao(1960). The proofs of our strong large deviation results depend heavily on the local limit theorems of Section 2. We shall develop some notation before stating the main theorem.

Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of nonlattice random variables with m.g.f. $\phi_n(z) = E[\exp(zT_n)]$, which is nonvanishing and analytic in the region $\Omega = \{z \in C : |z| < a\}$, where $a > 0$ and C is the set of all complex numbers. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \to \infty$. Let

(3-1) $$\psi_n(z) = a_n^{-1} \log \phi_n(z), \qquad \text{for } z \in \Omega, \text{ and}$$

(3-2) $$\gamma_n(u) = \sup_{|s|<a} [us - \psi_n(s)], \quad \text{for real } u.$$

Let $\{m_n, n \geq 1\}$ be a sequence of real numbers such that there exists a sequence $\{\tau_n\}$ satisfying $\psi_n'(\tau_n) = m_n$ and $d < \tau_n < a_1 < a$ for some positive numbers a_1, d and for all $n \geq 1$. The boundedness of τ_n below by $d > 0$ is satisfied for example if $\liminf_n[(m_n - E(T_n)/a_n)] > 0$. Theorem 3.1 below gives a strong large deviation result for T_n. One should note that condition(A) of Theorem 3.1 implies that $(T_n - E(T_n))/a_n$ converges to zero, in probability. Also, conditions (A) and (C) of Theorem 3.1 together

imply that $(T_n - E(T_n))/\sqrt{Var(T_n)}$ converges in distribution to the standard normal. A strong large deviation result for T_n when $\tau_n \to 0$ is proved later in Theorem 3.3. We now state the first theorem of this section.

Theorem 3.1. Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of nonlattice random variables. Let $\{m_n\}$ be a sequence of real numbers such that there exists $\{\tau_n\}$ satisfying $\psi_n'(\tau_n) = m_n$ and $d < \tau_n < a_1$, for all $n \geq 1$. Assume the following conditions for T_n:

(A) There exists $\beta < \infty$ such that $|\psi_n(z)| < \beta$ for all $n \geq 1,\ z \in \Omega$.

(B) There exists $\delta_1 > 0$ such that

$$\sup_{|t|\geq\delta} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right| = o\left(\frac{1}{\sqrt{a_n}}\right)$$

for all $0 < \delta < \delta_1$.

(C) There exists $\alpha > 0$ such that $\psi_n''(\tau_n) \geq \alpha$ for all $n \geq 1$.

Then

$$P\left(\frac{T_n}{a_n} \geq m_n\right) \sim \frac{\exp(-a_n\gamma_n(m_n))}{\sqrt{2\pi}\tau_n\sqrt{a_n\psi_n''(\tau_n)}}. \tag{3-3}$$

Proof. Let K_n be the distribution function of T_n. Let T_n^* be a random variable such that

$$P(T_n^* \leq y) = H_n(y) = \int_{-\infty}^{y} \exp(u\tau_n - a_n\psi_n(\tau_n))\, dK_n(u). \tag{3-4}$$

Let $T_n' = T_n^* - a_n m_n$. Then the c.f. of T_n' is given by

$$E(\exp(itT_n')) = \exp(-ita_n m_n)\frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)}. \tag{3-5}$$

Using these new random variables and the relation $\gamma_n(m_n) = m_n\tau_n - \psi_n(\tau_n)$, we have

$$\begin{aligned} P\left(\frac{T_n}{a_n} \geq m_n\right) &= \int_{a_n m_n}^{\infty} dK_n(y) \\ &= \int_{a_n m_n}^{\infty} \exp(-y\tau_n + a_n\psi_n(\tau_n))\, dH_n(y) \\ &= \exp(a_n\psi_n(\tau_n))\ E\big(\exp(-\tau_n T_n^*)\ I(T_n^* > a_n m_n)\big) \\ &= \exp(-a_n\gamma_n(m_n))\ E\big(\exp(-\tau_n T_n')I(T_n' \geq 0)\big) \\ &= \exp(-a_n\gamma_n(m_n))\ I_n \quad \text{(say)}. \end{aligned} \tag{3-6}$$

This step, usually called the use of Escher transformation, is the starting point of most investigations in large deviations. If the conditions (A), (B) and (C) are satisfied, Lemma 3.2 shows that Theorem 2.1 of the previous section holds for T'_n and for any $h > 0$,

$$\tau_n \sqrt{a_n \psi''_n(\tau_n)} P\big((k-1)h \le \tau_n T'_n < kh\big) \sim \frac{h}{\sqrt{2\pi}}, \tag{3-7}$$

uniformly for bounded intervals of k. Also, there exists constants M, n_h such that for $n \ge n_h$,

$$\Big|\tau_n \sqrt{a_n \psi''_n(\tau_n)}\; P\big((k-1)h \le \tau_n T'_n < kh\big)\Big| \le M \tag{3-8}$$

for all $k \ge 1$. We now write down lower and upper bounds for I_n:

$$\begin{aligned} I_n &= \sum_{k=1}^{\infty} E\big[\exp(-\tau_n T'_n)\; I\big((k-1)h \le \tau_n T'_n < kh\big)\big] \\ &\ge \sum_{k=1}^{k_h} \exp(-kh)\; P\big((k-1)h \le \tau_n T'_n < kh\big), \end{aligned} \tag{3-9}$$

and

$$\begin{aligned} I_n \le &\sum_{k=1}^{k_h} \exp(-(k-1)h)\; P\big((k-1)h \le \tau_n T'_n < kh\big) \\ &+ \sum_{k=k_h+1}^{\infty} \exp(-(k-1)h)\; P\big((k-1)h \le \tau_n T'_n < kh\big) \end{aligned} \tag{3-10}$$

where we choose $k_h = [1/h^2]$. Using (3-7) and (3-8) we get

$$\begin{aligned} \liminf_n \big[\sqrt{2\pi}\tau_n \sqrt{a_n \psi''_n(\tau_n)}\; I_n\big] &\ge \sum_{k=1}^{k_h} \exp(-kh)\, h \\ &= \frac{h\big(\exp(-h) - \exp(-(k_h+1)h)\big)}{1 - \exp(-h)} \end{aligned} \tag{3-11}$$

and

$$\begin{aligned} &\limsup_n \big[\sqrt{2\pi}\tau_n \sqrt{a_n \psi''_n(\tau_n)}\, I_n\big] \\ &\le \sum_{k=1}^{k_h} \exp(-(k-1)h)\, h + \sum_{k=k_h+1}^{\infty} M\sqrt{2\pi}\exp(-(k-1)h)\, h \\ &= \frac{h(1 - \exp(-k_h h))}{1 - \exp(-h)} + \frac{M\sqrt{2\pi}\exp(-k_h h)}{1 - \exp(-h)}. \end{aligned} \tag{3-12}$$

Letting $h \to 0$ we get from (3-11) and (3-12),

$$I_n \sim \frac{1}{\sqrt{2\pi}\tau_n\sqrt{a_n\psi_n''(\tau_n)}}. \tag{3-13}$$

This completes the proof of Theorem 3.1.◇

Lemma 3.2. Let $\{T_n', n \geq 1\}$ be a sequence of random variables as in Theorem 3.1. Assume that the conditions (A), (B) and (C) of Theorem 3.1 are satisfied. Then for any $h > 0$, (3-7) holds uniformly for bounded intervals of k. Further there exists constants M, n_h such that for $n \geq n_h$, (3-8) holds for all $k \geq 1$.

Proof. Let $d_n = \sqrt{a_n\psi_n''(\tau_n)}$. The lemma follows once we verify the conditions of Theorem 2.1 with $Y_n = T_n'/d_n$. The c.f. of Y_n is given by

$$\hat{f}_n(t) = \frac{\phi_n(\tau_n + it/d_n)}{\phi_n(\tau_n)} \exp(-itm_n a_n/d_n). \tag{3-14}$$

Since $\psi_n(z) = a_n^{-1}\log\phi_n(z)$ is a well defined and analytic function in Ω, and $|\tau_n| < a_1$, the following expansion is valid for $|t| < (a - a_1)/2$ and $n \geq 1$:

$$\psi_n(\tau_n + it) = \psi_n(\tau_n) + it\psi_n'(\tau_n) - (t^2/2)\psi_n''(\tau_n) + R_n(\tau_n + it). \tag{3-15}$$

Using condition (A) and Cauchy's theorem for derivatives we get for $|t| < (a - a_1)/2$,

$$|\psi_n^{(k)}(\tau_n + it)| \leq \frac{k!\,\beta}{(a - a_1)^k}, \quad \text{for } k \geq 1, \tag{3-16}$$

and

$$|R_n(\tau_n + it)| \leq \frac{2\beta|t|^3}{(a - a_1)^3}. \tag{3-17}$$

Therefore for $|t| < (a - a_1)/2$, we get from (3-15), (3-17) and condition (C),

(3-18)

$$\begin{aligned}
\log\hat{f}_n(t) &= -(itm_n a_n)/d_n + a_n\big[\psi_n(\tau_n + it/d_n) - \psi_n(\tau_n)\big] \\
&= -(itm_n a_n)/d_n + a_n\big[it\psi_n'(\tau_n)/d_n - (t^2\psi_n''(\tau_n))/(2d_n^2) + R_n(\tau_n + it/d_n)\big] \\
&= -t^2/2 + a_n R_n(\tau_n + it/d_n),
\end{aligned}$$

and

$$\text{(3-19)} \qquad |a_n R_n(\tau_n + it/d_n)| \leq \frac{2\beta|t|^3}{\alpha d_n(a-a_1)^3} \to 0, \quad \text{as } n \to \infty.$$

Hence $Y_n = T_n'/d_n$, converges in distribution to the standard normal random variable. We now proceed to verify conditions (2-1) and (2-2) of Theorem 2.1. Let

$$\text{(3-20)} \qquad \begin{aligned} g_n(t) &= d_n^{-2}\log|\hat{f}_n(d_n t)| \\ &= \frac{-itm_n}{\psi_n''(\tau_n)} + \frac{1}{\psi_n''(\tau_n)}\left[\mathrm{Real}(\psi_n(\tau_n+it)) - \psi_n(\tau_n)\right]. \end{aligned}$$

Thus

$$\text{(3-21)} \qquad \begin{aligned} g_n''(t) &= -\frac{\mathrm{Real}(\psi_n''(\tau_n+it))}{\psi_n''(\tau_n)} \\ &= -\frac{\mathrm{Real}(\psi_n''(\tau_n)+it\theta_n^*)}{\psi_n''(\tau_n)} \\ &= -1 + \mathrm{Real}(it\theta_n^*/\psi_n''(\tau_n)) \\ &\leq -1 + |t||\theta_n^*|/\alpha, \end{aligned}$$

where θ_n^* is an appropriate complex number. By (3-16) we get that

$$\text{(3-22)} \qquad |\theta_n^*| \leq \frac{3!\beta}{(a-a_1)^3} \quad \text{for } n \geq 1.$$

Therefore we can find $\delta > 0$ such that for $|t| < \delta$,

$$\text{(3-23)} \qquad g_n''(t) \leq -(1/2) \qquad \text{for all } n \geq 1.$$

This verifies condition (2-1) of Theorem 2.1 with d_n replaced by δd_n as noted in Theorem 2.5. Now, from condition (B) we get that

$$\text{(3-24)} \qquad \begin{aligned} \sup_{|t|\geq \delta d_n} |\hat{f}_n(t)| &= \sup_{|t|\geq\delta}\left|\frac{\phi_n(\tau_n+it)}{\phi_n(\tau_n)}\right| \\ &= o\left[\frac{1}{\sqrt{a_n}}\right] \\ &= o\left[\frac{1}{\tau_n d_n}\right] \end{aligned}$$

since τ_n is bounded and $d_n = O(\sqrt{a_n})$. This verifies condition (2-2) of Theorem 2.1 with $b_n = \tau_n d_n$. The assertions (3-7) and (3-8) now follow from (2-3) and (2-4) respectively. ◇

We now turn our attention to the case where $\tau_n \to 0$ as $n \to \infty$, but not very fast. More specifically, we require that $\tau_n\sqrt{a_n} \to \infty$. In this case we can get the stronger result that the conclusion of Theorem 3.1 holds without condition (B).

Theorem 3.3. Let $\{T_n,\ n \geq 1\}$ be an arbitrary sequence of nonlattice random variables. Let $\{m_n\}$ be a sequence of real numbers such that there exists a sequence $\{\tau_n\}$ satisfying $\psi_n'(\tau_n) = m_n$, $\tau_n > 0$. Also assume that $\tau_n \to 0$ and $\tau_n\sqrt{a_n} \to \infty$. Let T_n satisfy the conditions (A) and (C) of Theorem 3.1. Then

$$P\left(\frac{T_n}{a_n} \geq m_n\right) \sim \frac{\exp(-a_n\gamma_n(m_n))}{\sqrt{2\pi}\tau_n\sqrt{a_n\psi_n''(\tau_n)}}. \tag{3-25}$$

The proof of Theorem 3.3 is similar to the proof of Theorem 3.1. The only change is that we apply Lemma 3.4 instead of Lemma 3.2 to obtain (3-7) and (3-8).

Lemma 3.4. Let $\{T_n',\ n \geq 1\}$ be a sequence of random variables as defined in the proof of Theorem 3.1. Let $\tau_n \to 0$ and $\tau_n\sqrt{a_n} \to \infty$. Assume that conditions (A) and (C) of Theorem 3.1 are satisfied. Then for any $h > 0$, (3-7) holds for bounded intervals of k. Further, there exists constants M, n_h such that for $n \geq n_h$, (3-8) holds uniformly for all $k \geq 1$.

Proof. As before let $Y_n = T_n'/d_n$, where $d_n = \sqrt{a_n\psi_n''(\tau_n)}$. We have already seen that in Lemma 3.2, Y_n converges in distribution to standard normal random variable if conditions (A) and (C) are satisfied. Also, Y_n satisfies condition (2-1) of Theorem 2.1. Let $b_n = \tau_n d_n$. The assumptions on τ_n imply that $b_n \to \infty$ and $d_n/b_n \to \infty$ as $n \to \infty$. Therefore the conclusions (2-3) and (2-4) are valid for Y_n by Theorem 2.4. This proves Lemma 3.4. ◇

4. The Lattice Case

This section primarily deals with local limit theorems and strong large deviation theorems for lattice valued random variables. These theorems are analogous to the theorems for nonlattice valued random variables of the previous two sections.

Theorem 4.1. Let Y_n be a lattice valued random variable taking values in the lattice $\{c_n + kh_n : k = 0, \pm 1, \pm 2, \ldots\}$, where $h_n > 0$ and $n \geq 1$. Assume that the span h_n of Y_n, converges to zero as $n \to \infty$. Let Y_n converge in distribution to Y. Let $\hat{f}_n$ be the c.f. of Y_n and $\hat{f}$ be the c.f. of Y. Let $\{d_n\}$ be a sequence of real numbers with $d_n \to \infty$. Assume that there exists an integrable function f^* such that

$$\sup_n |\hat{f}_n(t)| \, I(|t| < d_n) \leq f^*(t) \tag{4-1}$$

for each t, and

$$\sup_{d_n \leq |t| < \pi/h_n} |\hat{f}_n(t)| = \theta_n = o(h_n), \quad \text{as } n \to \infty. \tag{4-2}$$

Let y_n be in the range of Y_n, such that y_n converges to y^*, as $n \to \infty$. Then

$$\frac{1}{h_n} P(Y_n = y_n) \to f(y^*), \tag{4-3}$$

where f is the p.d.f. of Y. Also, there exists a constant $M < \infty$, and n_0 such that for $n \geq n_0$,

$$\left[\frac{1}{h_n} P(Y_n = y)\right] \leq M \tag{4-4}$$

uniformly in y.

Proof. Let y_n is a possible value of Y_n. Then an application of the inversion formula yields

(4-5)
$$\begin{aligned}\frac{1}{h_n}P(Y_n = y_n) &= \frac{1}{2\pi}\int_{-\pi/h_n}^{\pi/h_n} \hat{f}_n(t)\exp(-ity_n)\,dt \\ &= \frac{1}{2\pi}\int_{|t|<d_n} \hat{f}_n(t)\exp(-ity_n)\,dt + \frac{1}{2\pi}\int_{d_n\le|t|<\pi/h_n} \hat{f}_n(t)\exp(-ity_n)\,dt \\ &= I_{n1} + I_{n2} \qquad \text{(say)}.\end{aligned}$$

It is easy to check that condition (4-1) and Dominated convergence theorem imply that I_{n1} converges to $f(y^*) = (1/2\pi)\int \hat{f}(t)\exp(-ity^*)\,dt$. Next

$$\begin{aligned}|I_{n2}| &\le \frac{1}{h_n}\sup_{d_n\le|t|<\pi/h_n}|\hat{f}_n(t)| \\ &= \frac{\theta_n}{h_n},\end{aligned} \tag{4-6}$$

which converges to zero as $n \to \infty$, by condition (4-2). This completes the proof of (4-3). Next, from (4-5) and (4-6) we get

$$\begin{aligned}\left|\frac{1}{h_n}P(Y_n = y)\right| &\le \frac{1}{2\pi}\int_{|t|<d_n}|\hat{f}_n(t)|\,dt + \frac{\theta_n}{h_n} \\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}|f^*(t)|\,dt + \frac{\theta_n}{h_n} \\ &\le \frac{1}{\pi}\int_{-\infty}^{\infty}|f^*(t)|\,dt = M,\end{aligned} \tag{4-7}$$

uniformly in y, for sufficiently large $n \ge n_0$. This completes the proof of the theorem. ◇

Remark 4.2. In the above Theorem 4.1, it is assumed that $d_n < \pi/h_n$ for all $n \ge 1$. Suppose on the contrary that $d_n \ge \pi/h_n$ for all $n \ge n_1$. The above proof shows that the conclusions (4-3) and (4-4) hold. The condition (4-2) becomes vacuous and should be ignored.

The next theorem provides an estimate of the large deviation probability for arbitrary sequence $\{T_n, n \ge 1\}$ of lattice valued random variables. We begin with some preliminaries.

Let $\{T_n,\, n \geq 1\}$ be an arbitrary sequence of lattice valued random variables taking values in the lattice $\{t_n + kp_n : k = 0, \pm 1, \pm 2, \ldots\}$, $p_n > 0$. Let the c.f. of T_n, $\phi(z)$, be analytic and nonvanishing in the region $\Omega = \{z \in C : |z| < a\}$. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \to \infty$ and $p_n = o(\sqrt{a_n})$. Let

$$\psi_n(z) = a_n^{-1} \log \phi_n(z), \tag{4-8}$$

be a well defined analytic function on Ω. Let $\{m_n\}$ be a sequence of real numbers contained in the range of T_n/a_n, such that there exists $0 < \tau_n < a_1 < a$ satisfying $\psi_n'(\tau_n) = m_n$ and $\tau_n\sqrt{a_n} \to \infty$ as $n \to \infty$. With this notation we are ready to state our next theorem.

Theorem 4.3. Assume that T_n satisfies conditions (A), (C) of Theorem 3.1 and the following condition (B′):

(B′) There exists $\delta_1 > 0$, such that for $0 < \delta < \delta_1$,

$$\sup_{\delta \leq |t| < \pi/p_n} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right| = o\left[\frac{p_n}{\sqrt{a_n}}\right].$$

Then

$$P\left(\frac{T_n}{a_n} \geq m_n\right) \sim \frac{p_n}{\sqrt{2\pi}\sqrt{a_n\psi_n''(\tau_n)}} \frac{\exp(-a_n\gamma_n(m_n))}{(1 - \exp(-p_n\tau_n))} \tag{4-9}$$

where $\gamma_n(m_n) = m_n\tau_n - \psi_n(\tau_n)$.

Proof. Since m_n is in the range of T_n/a_n, we can write $a_n m_n = t_n + l_n p_n$ for some integer l_n. Consider

$$\begin{aligned} P\left(\frac{T_n}{a_n} \geq m_n\right) &= P(T_n \geq t_n + l_n p_n) \\ &= \sum_{k=l_n}^{\infty} P(T_n = t_n + kp_n) \\ &= \exp(-a_n\gamma_n(m_n)) \sum_{k=l_n}^{\infty} \exp(a_n\gamma_n(m_n)) P(T_n = t_n + kp_n) \\ &= \exp(-a_n\gamma_n(m_n)) \sum_{k=l_n}^{\infty} \exp(-(k - l_n)p_n\tau_n) P_n(k) \end{aligned} \tag{4-10}$$

where

$$P_n(k) = \frac{\exp((t_n + kp_n)\tau_n)}{\phi_n(\tau_n)} P(T_n = t_n + kp_n). \tag{4-11}$$

Let us introduce for each $n \geq 1$ a lattice valued random variable T_n' which takes the value $(k - l_n)p_n$ with probability $P_n(k)$. Therefore, we can rewrite (4-10) as

$$\begin{aligned} P\left(\frac{T_n}{a_n} \geq m_n\right) &= \exp(-a_n\gamma_n(m_n))\, E\big(\exp(-\tau_n T_n')\, I(T_n' \geq 0)\big) \\ &= \exp(-a_n\gamma_n(m_n))\, I_n \qquad \text{(say)}. \end{aligned} \tag{4-12}$$

Let $d_n = \sqrt{a_n\psi_n''(\tau_n)}$ and $Y_n = T_n'/d_n$. Then Y_n is a lattice valued random variable with span $h_n = p_n/d_n$. Note that $h_n \to 0$ as $n \to \infty$. If the conditions (A), (B′) and (C) are satisfied, the next Lemma 4.4 shows that Y_n converges in distribution to the standard normal and satisfies the hypothesis of Theorem 4.1. Thus we obtain

$$\frac{1}{h_n} P\big(Y_n = kh_n\big) \to \frac{1}{\sqrt{2\pi}}, \quad \text{as } n \to \infty, \tag{4-13}$$

uniformly in $k \in [-k_n, k_n]$, where $k_n h_n \to 0$. Also, there exists $M < \infty$ and n_0 such that for $n \geq n_0$,

$$\left[\frac{1}{h_n} P\big(Y_n = kh_n\big)\right] \leq M \tag{4-14}$$

for all k. We are now in a position to evaluate the expectation on the r.h.s. of (4-12). Consider

$$\begin{aligned} I_n &= E\big(\exp(-d_n\tau_n Y_n) I(Y_n \geq 0)\big) \\ &= \sum_{k=0}^{\infty} \exp(-kp_n\tau_n) P\big(Y_n = kh_n\big). \end{aligned} \tag{4-15}$$

Let $k_n = [d_n/\tau_n p_n^2]^{1/2}$. Note that $k_n h_n \to 0$ and $k_n p_n \tau_n \to \infty$ since $\tau_n\sqrt{a_n} \to \infty$. A lower bound for the r.h.s. of (4-15) is

$$\sum_{k=0}^{k_n} \exp(-kp_n\tau_n)\, P\big(Y_n = kh_n\big) \tag{4-16}$$

and an upper bound is given by

$$(4\text{-}17)\qquad \sum_{k=0}^{k_n} \exp(-kp_n\tau_n)P(Y_n = kh_n) + Mh_n \sum_{k=k_n+1}^{\infty} \exp(-kp_n\tau_n)$$

wherein we have used (4-14). Combining (4-15), (4-16) and (4-17) we get

$$(4\text{-}18)\qquad \begin{aligned} \liminf_n \frac{(1-\exp(-p_n\tau_n))}{h_n} I_n &\geq \frac{1}{\sqrt{2\pi}} \liminf_n (1-\exp(-k_n p_n\tau_n)) \\ &= \frac{1}{\sqrt{2\pi}} \end{aligned}$$

and

$$(4\text{-}19)\qquad \begin{aligned} \limsup_n \frac{(1-\exp(-p_n\tau_n))}{h_n} I_n &\leq \frac{1}{\sqrt{2\pi}} + \limsup_n (M\exp(-k_n p_n\tau_n)) \\ &= \frac{1}{\sqrt{2\pi}} \end{aligned}$$

since $k_n p_n \tau_n \to \infty$ as $n \to \infty$. Therefore

$$(4\text{-}20)\qquad I_n \sim \frac{h_n}{(1-\exp(-p_n\tau_n))\sqrt{2\pi}}.$$

The proof of the theorem is completed substituting (4-20) in (4-12). $\diamond$

We now state and prove Lemma 4.4 which was used in a major way in the proof of the above theorem.

Lemma 4.4. Let Y_n be a lattice valued random variable taking values in the lattice $\{(k-l_n)h_n : k = 0, \pm1, \pm2, \ldots\}$, with probabilities $\{P_n(k) : k = 0, \pm1, \pm2, \ldots\}$, where P_n is given by (4-11). Let $h_n \to 0$ as $n \to \infty$. If the conditions (A), (B′) and (C) are satisfied then Y_n converges in distribution to standard normal. Furthermore, (4-13) holds uniformly in $k \in [-k_n, k_n]$, where $k_n h_n \to 0$ and the inequality (4-14) holds for all k.

Proof. The lemma will be proved once we verify that Y_n satisfies the conditions of Theorem

4.1. The c.f. of Y_n is given by

$$
\begin{aligned}
\hat{f}_n(t) &= E\big(\exp(itY_n)\big) \\
&= \sum_{k=-\infty}^{\infty} \exp(it(k-l_n)h_n)P_n(k) \\
&= \sum_{k=-\infty}^{\infty} \exp(it(k-l_n)h_n + (t_n + kp_n)\tau_n)\frac{P(T_n = t_n + kp_n)}{\phi_n(\tau_n)} \qquad (4\text{-}21) \\
&= \exp(-itm_n a_n/d_n)\frac{\phi_n(\tau_n + it/d_n)}{\phi_n(\tau_n)},
\end{aligned}
$$

wherein we have used the fact $a_n m_n = t_n + l_n p_n$. As in Lemma 3.2, we can show that $\hat{f}_n(t)$ converges to $\exp(-t^2/2)$ and hence Y_n converges in distribution to standard normal. Imitating the proof of Lemma 3.2, we can also show that if conditions (A), (B′) and (C) are satisfied then $\hat{f}_n(t)$ satisfies the conditions of Theorem 4.1. The rest of the Lemma 4.4 follows from Theorem 4.1. ◇

5. Applications

In this section we give two typical applications to illustrate the large deviation limit theorems and strong large deviation limit theorems of the previous sections. The first example is a local limit result and illustrates Theorem 2.1. The second example is a strong large deviation result for a lattice valued random variable and illustrates Theorem 4.3.

Example 5.1. This example applies to a general class of sums of dependent random variables considered in Chaganty and Sethuraman(1986a). Though it was proved in that paper that the limit distribution could be both normal and nonnormal, our example applies only to the case where the limit distribution is normal. We first present a particular application and then state a more general application referring to conditions found in Chaganty and Sethuraman(1986a).

Let $\{X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}\}$ be a triangular array of random variables with joint density function

$$dQ_n(\mathbf{x}) = z_n^{-1}(2\pi)^{-n/2}\left[\cosh(s_n/\sqrt{2}n)\right]^n \exp(-\sum_{j=1}^{n} x_j^2/2)\, d\mathbf{x}, \tag{5-1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $s_n = x_1 + \ldots + x_n$ and z_n is a normalizing constant. Such dependent random variables arise in generalized Curie-Weiss models used to describe ferromagnets. The constant $\sqrt{2}$ inside the argument of the cosh function above plays an important role. Example 4.4 of Chaganty and Sethuraman(1986a) can be modified or Theorem 3.7 of that paper can be used directly to show that $Y_n = (X_1^{(n)} + \ldots + X_n^{(n)})/\sqrt{n}$ converges in distribution to a normal distribution with mean 0 and variance 2 (Example 4.4 of Chaganty and Sethuraman(1986a) used the constant 1 instead of $\sqrt{2}$ and obtained a non-

normal distribution under a different normalization). We will now show that Theorem 2.1 applies to Y_n. Since

$$(5\text{-}2) \qquad (\cosh \omega)^n = \sum_{y \in C_n} \exp(\omega y)\lambda_n(y)$$

with $\lambda_n(y) = \binom{n}{\frac{n+y}{2}}$ and $C_n = \{-n, -n+2, \ldots, n\}$, the c.f. of Y_n is given by

$$\begin{aligned}(5\text{-}3) \quad \hat{f}_n(t) &= E(\exp(itY_n)) \\ &= z_n^{-1} \sum_{y \in C_n} \left[\frac{1}{(2\pi)^{n/2}} \int \exp(its_n/\sqrt{n} + ys_n/\sqrt{2n} - \sum_{j=1}^{n} x_j^2/2) dx \right] \lambda_n(y) \\ &= \exp(-t^2/2) z_n^{-1} \sum_{y \in C_n} \exp(ity/\sqrt{2n} + y^2/4n)\lambda_n(y).\end{aligned}$$

Since $\hat{f}_n(0) = 1$, we have

$$(5\text{-}4) \qquad |\hat{f}_n(t)| \leq \exp(-t^2/2) \qquad \text{for all } n \text{ and } t.$$

Thus from Theorem 2.1 and Remark 2.2 it follows that for any $h > 0$, $\{b_n\} \to \infty$ and $y_n \to y$,

$$(5\text{-}5) \qquad b_n P\left(\left| \frac{S_n}{\sqrt{n}} - y_n \right| < \frac{h}{b_n} \right) \to \frac{h}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2} \right)$$

with $\sigma = \sqrt{2}$.

From the above discussion and from a full use of Theorem 3.7 of Chaganty and Sethuraman(1985a) we have the following application which we state without proof.

Let $\{X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}\}$ be a triangular array of random variables whose joint distribution is as given in (3.13) of Theorem 3.7 of Chaganty and Sethuraman(1985a). We will impose conditions on the probability measure P and the index r appearing in that Theorem. Let P be the standard normal distribution and let $r = 1$. Under these conditions, Theorem 3.7 of Chaganty and Sethuraman(1985a) shows that there is a sequence of constants $\{m_n\}$ such that

$$(5\text{-}6) \qquad Y_n = (\sum_{j=1}^{n} X_i^{(n)} - nm_n)/\sqrt{n}$$

has a limiting normal distribution with mean 0 and variance σ^2. Let $\hat{f}_n(t)$ be the c.f. of Y_n. For this case, if we proceed as in the application above, we can establish (5-5) for all n and t. This shows that (5-5) is true with the appropriate σ.

Example 5.2. We now obtain a strong large deviation result for the Wilcoxon signed-rank statistic under the null hypothesis. This stregthens the well known weak large deviation results for this statistic (see Klotz(1965)).

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with median m. Arrange $|X_1|, |X_2|, \ldots, |X_n|$ in increasing order of magnitude and assign ranks $1, 2, \ldots, n$. The Wilcoxon signed-rank statistic U_n is defined as the sum of the ranks of positive X_i's. The statistic U_n is used to test the null hypothesis $H_0 : m = 0$ vs $H_1 : m \neq 0$. Let $T_n = U_n/n$. The random variable T_n is a lattice random variable with span $p_n = 1/n$. The c.f. of T_n under the null hypothesis H_0 is given by

(5-7) $$\phi_n(z) = \prod_{k=1}^{n} \left[(\exp(kz/n) + 1)/2\right], \qquad z \in C.$$

It is easy to check that $\phi_n(z)$ is analytic and nonvanishing in the region $\Omega = \{z \in C : |z| < \pi/2\}$. Let

(5-8) $$\psi_n(z) = n^{-1} \log \phi_n(z).$$

We will verify that T_n satisfies all the conditions of Theorem 4.3. It is easy to check that there exists $\beta > 0$ such that $|\psi_n(z)| < \beta$ for $|z| < \pi/2$. Straightforward calculations show that $\psi_n''(\tau)$ is bounded below by a positive number α for real τ such that $|\tau| < \pi/2$. Thus T_n satisfies conditions (A) and (C). Now to verify condition (B$'$) we first note that the range of $\psi_n'(s)$, for real s contains the open interval (0, 1/2) for all $n \geq 1$. Thus if $\{m_n\}$ is a sequence of real numbers contained in a proper subinterval of (1/4, 1/2) then we can find positive numbers d, a_1 and a sequence $\{\tau_n\}$ such that $d < \tau_n < a_1 < \pi/2$ and $\psi_n'(\tau_n) = m_n$ for all $n \geq 1$. Therefore $\sqrt{n}\tau_n \to \infty$ as $n \to \infty$. From the analysis in

Example 3.1 of Chaganty and Sethuraman(1985) it can be seen that there exists n_0 and $\delta_1 > 0$ such that for $0 < \delta < \delta_1$,

$$\text{(5-9)} \qquad \sup_{\delta \leq |t| < \pi/p_n} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right| \leq \exp(-n\alpha\delta^2/4)$$

for $n \geq n_0$. This verifies condition (B′). Therefore the conclusion (4-9) of Theorem 4.3 holds and it provides an asymptotic expression for $P(T_n \geq n\, m_n)$.

References

1. Bahadur, R.R. and Ranga Rao, R.(1960). On deviations of the sample mean. *Ann. Math. Statist.*, **31** 1015-1027.

2. Chaganty, N.R. and Sethuraman, J.(1985). Large deviation local limit theorems for arbitrary sequences of random variables. *Ann. of Probab.*, **13** 97-114.

3. Chaganty, N.R. and Sethuraman, J.(1986a). Limit theorems in the area of large deviations for some dependent random variables. To appear in *Ann. of Probab.*

4. Chaganty, N.R. and Sethuraman, J.(1986b). Multi-dimensional large deviation local limit theorems. To appear in *Journal of Multi. Analysis.*

5. Chernoff, H.(1952). A measure of asymptotic efficiency for tests of an hypothesis based on the sum of observations. *Ann. Math. Statist.*, **23** 493-507.

6. Cramér, H.(1938). Sur un nouveau théorème-limite de la théorie des probabilitès. *Actualites Sci. Ind.* **736** 5-23.

7. DeHaan, L. and Resnick, S.I.(1982). Local limit theorems for sample extremes. *Ann. of Probab.*, **2** 396-413.

8. Ellis, R.S.(1984). Large deviations for a general class of dependent random vectors, *Ann. of probab.*, **12** 1-12.

9. Feller, W.(1967). On regular variation and local limit theorems. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.*, Vol II, Part I, 373-388. University of California Press, Berkeley.

10. Jain, N.C. and Pruitt, W.E.(1985). Lower tail probability estimates for subordinators and nondecreasing random walks. Preprint.

11. Klotz, J.(1965). Alternative efficiencies for signed rank tests. *Ann. Math. Statist.*, **36** 1759-1766.

12. Sievers, G.L.(1969). On the probability of large deviations. *Ann. of Statist.*, **40** 1908-1921.

13. Steinebach, J.(1978). Convergence rates of large deviation probabilities in the multidimensional case. *Ann. of Probab.*, **5** 751-759.

14. Varadhan, S.R.S. (1984). *Large deviations and Applications.* SIAM, CBMS/NSF Regional Conference in Applied Math., **46** SIAM, philadelphia.

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Most large deviation results give asymptotic expressions to $\log P(Y_n \geq x_n)$ where the event $\{Y_n \geq x_n\}$ is a large deviation event, that is, its probability goes to zero exponentially fast. We refer to such results as weak large deviation results. In this paper we obtain strong large deviation results for arbitrary random variables $\{Y_n\}$, that is, we obtain asymptotic expressions for $P(Y_n \geq x_n)$

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where $\{Y_n \geq x_n\}$ is a large deviation event. These strong large deviation results are obtained for lattice valued and nonlattice valued random variables and require some conditions on their moment generating functions.

A result that gives the limit of the average probability that Y_n lies in an interval $2h/b_n$ around the point Y_n, where $h > 0$, $b_n \to \infty$ and $y_n \to y^*$, is referred to as a local limit result for $\{Y_n\}$. In this paper we obtain local limit theorems for arbitrary random variables based on easily verifiable conditions on their characteristic functions. These local limit theorems play a major role in the proofs of the strong large deviation results of this paper.

We illustrate these results with two typical applications.